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TECHNICAL NOTE

Inverse analysis of transient turbulent forced convection inside parallel-plate ducts

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INTRODUCTION

Transient convective heat transfer inside ducts with timewise variation of wall heat flux is of interest in the control of heat exchanger equipment and most of the work available in the literature deals with the solution of direct problems [1–3, 6]. However, there are applications in which boundary conditions or thermophysical properties for the problem are unknown, but transient temperature readings taken at a specific location are available as a function of time. Then the problem of estimating, say, the unknown wall heat flux, by utilizing the available experimental data, is an inverse problem.

Available work in the inverse heat convection problems is still very limited [4, 5]. The purpose of this work is twofold: first to demonstrate the feasibility of the conjugate gradient method with an adjoint equation for solving the inverse turbulent forced convection problem of estimating the timewise varying wall heat flux by using simulated measured data, with no prior information on the functional form of the unknown heat flux. Second, to examine the effects of sensor location, magnitude of the measurement error, functional form of the timewise variation of heat flux on the accuracy of estimations.

DIRECT PROBLEM

We consider hydrodynamically developed, thermally developing transient heat transfer for an incompressible, turbulent, constant property flow inside a parallel-plate duct subjected to timewise varying wall heat flux at both boundaries. Axial conduction, viscous dissipation, free convection and wall conjugation effects are neglected. Because of symmetry, only half the region is considered. The mathematical formulation of this problem in the dimensionless form is given by

$$\frac{\partial\Theta(x,y,\tau)}{\partial\tau}+U(y)\frac{\partial\Theta(x,y,\tau)}{\partial x}=\frac{\partial}{\partial y}\left\{\varepsilon_t(y)\frac{\partial\Theta(x,y,\tau)}{\partial y}\right\} \tag{1a}$$

$$\frac{\partial\Theta}{\partial y}=Q(\tau)\quad\text{at}\quad y=1;\quad \frac{\partial\Theta}{\partial y}=0\quad\text{at}\quad y=0 \tag{1b,c}$$

$$\Theta=1\quad\text{at}\quad x=0;\quad \Theta=1\quad\text{at}\quad \tau=0. \tag{1d,e}$$

Here, the fully developed turbulent velocity distribution, $U(y)$, and the total diffusivity, ε_t, are determined by a turbulent model given in the Appendix of ref. [6].

THE INVERSE PROBLEM

The inverse problem considered here is concerned with the estimation of the unknown boundary heat flux $Q(\tau)$ from the knowledge of transient temperature readings taken at a specified location as a function of time. To solve such an inverse problem we are concerned with the minimization of the residual functional $J(Q)$ defined as

$$J(Q)=\int_0^{\tau_f}(\Theta-Z)^2\,d\tau, \tag{2}$$

where Θ is the temperature computed from the solutions of direct problem defined by equations (1) by using the estimate for Q; and Z is the measured temperature at the sensor location ($x = x^*$, $y = y^*$). Thus, the inverse problem is recast into an optimum control problem, i.e. to find the wall heat flux (Q), which minimizes the residual functional, J.

To solve this optimization problem by using the conjugate gradient method of minimization, we need to construct: (i) the sensitivity problem, (ii) the adjoint problem, and (iii) the gradient equation, which are then used in the minimization procedure as described below.

THE SENSITIVITY PROBLEM

When the wall heat flux $Q(\tau)$ undergoes an increment change $\Delta Q(\tau)$, the temperature $\Theta(x,y,\tau)$ also changes by the amounts $\Delta\Theta(x,y,\tau)$. To construct the sensitivity problem satisfying the functions $\Delta\Theta$ and ΔQ, we replace Θ and Q in the direct problem (1a–e) by $\Theta+\Delta\Theta$ and $Q+\Delta Q$, respectively, and then subtract from the resulting equations the original direct problem. The following sensitivity problem is obtained:

$$\frac{\partial\,\Delta\Theta(x,y,\tau)}{\partial\tau}+U(y)\frac{\partial\Delta\Theta(x,y,\tau)}{\partial x}=\frac{\partial}{\partial y}\left\{\varepsilon_t(y)\frac{\partial\,\Delta\Theta(x,y,\tau)}{\partial y}\right\} \tag{3a}$$

$$\frac{\partial\,\Delta\Theta}{\partial y}=\Delta Q(\tau)\quad\text{at}\quad y=1;\quad \frac{\partial\,\Delta\Theta}{\partial y}=0\quad\text{at}\quad y=0 \tag{3b,c}$$

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NOMENCLATURE

$\bar{D}_e$	equivalent diameter
$\bar{h}$	dimensional half channel height
J	residual functional defined by equation (2)
J'	gradient of the function J
K	thermal conductivity
k	kth iteration
L	$= \bar{L}/(Re\,Pr\,\bar{D}_e/16)$; dimensionless channel length
$\bar{L}$	dimensional channel length
n	number of sensors
P	direction of descent defined by equation (11)
Pr	Prandtl number
$\bar{q}$	dimensional heat flux
Q	$= \bar{q}/(T_0 K/\bar{h})$; dimensionless heat flux
Re	$= u_m \bar{D}_e/\nu$; Reynolds number
t	dimensional time
T	dimensional computer temperature
T_0	dimensional inlet temperature
u_m	mean flow velocity
$u(\bar{y})$	dimensional velocity
$U(y)$	$= u/u_m$; dimensionless velocity
x	$= (16\bar{x}/\bar{D}_e)/Re\,Pr$; dimensionless axial distance along the channel
$\bar{x}$	dimensional axial distance along the channel
x^*	$= \bar{x}^*/(Re\,Pr\,\bar{D}_e/16)$; dimensionless axial location of sensor
$\bar{x}^*$	dimensional axial location of sensor
y	$= \bar{y}/\bar{h}$; dimensionless transverse coordinate
$\bar{y}$	dimensional transverse coordinate
y^*	$= \bar{y}^*/\bar{h}$; dimensionless transverse location of sensor
$\bar{y}^*$	dimensional transverse location of sensor
$Z(x, y, \tau)$	dimensionless measured temperature.

Greek symbols

α	thermal diffusivity
β	step size in going from k to $k+1$ iteration in equation (10)
γ	conjugate coefficient, defined by equations (11) and (12)
δ	Dirac delta function
ξ	convergence criteria
ε_h	eddy diffusivity for heat
ε_t	total diffusivity
ρ	fluid density
$\lambda(x, y, \tau)$	adjoint function satisfying the adjoint problem defined by equations (6)
τ	$= \alpha t/\bar{h}^2$; dimensionless time
$\Theta(x, y, \tau)$	$= T/T_0$; dimensionless computed temperature
σ	standard deviation of temperature
ω	random number.

$$\Delta\Theta = 0 \quad \text{at} \quad x = 0; \quad \Delta\Theta = 0 \quad \text{at} \quad \tau = 0. \tag{3d,e}$$

THE ADJOINT PROBLEM

To derive the adjoint equations we multiply equation (1a) by the adjoint function $\lambda(x, y, \tau)$, integrate the resulting expression over the space and time domains and then add the resulting expression to equation (2) to yield

$$J = \int_0^{\tau_f} (\Theta - Z)^2 \, d\tau + \int_0^{\tau_f}\int_0^L\int_0^1 \lambda\left\{\frac{\partial}{\partial y}\left(\varepsilon_t \frac{\partial\Theta}{\partial y}\right) - \frac{\partial\Theta}{\partial\tau} - U\frac{\partial\Theta}{\partial x}\right\} dy\,dx\,d\tau. \tag{4}$$

To obtain the variation of J, Θ is perturbed by $\Delta\Theta$, and then the equation (4) is subtracted from it

$$\Delta J = \int_0^{\tau_f}\int_0^L\int_0^1 2(\Theta - Z)\,\Delta\Theta \times \delta(x - x^*)\,\delta(y - y^*)\,dy\,dx\,d\tau + \int_0^{\tau_f}\int_0^L\int_0^1 \lambda\left\{\frac{\partial}{\partial y}\left(\varepsilon_t\frac{\partial\Delta\Theta}{\partial y}\right) - \frac{\partial\Delta\Theta}{\partial\tau} - U\frac{\partial\Delta\Theta}{\partial x}\right\} dy\,dx\,d\tau. \tag{5}$$

The second integral term on the right-hand side is simplified by integration by parts and applying the initial and boundary conditions from sensitivity problem (3). After some manipulation and rearrangement we find the following adjoint problem for the determination of the adjoint function $\lambda(x, y, \tau)$

$$\frac{\partial\lambda}{\partial\tau} + U\frac{\partial\lambda}{\partial x} + \frac{\partial}{\partial y}\left(\varepsilon_t\frac{\partial\lambda}{\partial y}\right) + 2(\Theta - Z) \times \delta(x - x^*)\,\delta(y - y^*) = 0 \tag{6a}$$

$$\frac{\partial\lambda}{\partial y} = 0 \quad \text{at} \quad y = 1; \quad \frac{\partial\lambda}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{6b,c}$$

$$\lambda = 0 \quad \text{at} \quad x = L; \quad \lambda = 0 \quad \text{at} \quad \tau = \tau_f \tag{6d,e}$$

and the remaining term gives the ΔJ as

$$\Delta J = -\int_0^{\tau_f}\int_0^L \lambda(x, 1, \tau)\varepsilon_t(1)\,dx\,d\tau. \tag{7}$$

THE GRADIENT EQUATION

The gradient J' of the function J is defined by [7]

$$\Delta J = \int_0^{\tau_f} J'(\tau)\,\Delta Q\,d\tau. \tag{8}$$

Comparing equations (7) and (8), we conclude that the gradient equation J' is given by

$$J'(\tau) = -\int_0^L \lambda(x, 1, \tau)\varepsilon_t(1)\,dx. \tag{9}$$

MINIMIZATION PROCEDURE

The iterative procedure for determination of the wall heat flux is computed from [8]

$$Q^{k+1} = Q^k - \beta^k P^k, \tag{10}$$

where P^k is the direction of descent, defined as the combination of gradient at step k and the descent direction at step $k-1$, in the form

$$P^k = J'^k + \gamma^k P^{k-1}. \tag{11}$$

Different definitions of the conjugate coefficient γ can be found in the standard texts of mathematics [7], here we choose the form

$$\gamma^k = \left(\int_0^{\tau_f} [J'^k(\tau)]^2 \, d\tau\right) \bigg/ \left(\int_0^{\tau_f} [J'^{k-1}(\tau)]^2 \, d\tau\right) \quad \text{with } \gamma^0 = 0. \tag{12}$$

The coefficient β^k, which determines the step size in going from k to $k+1$, is determined by minimizing J in equation (2) with respect to β^k, i.e.

$$\min_\beta J(Q^{k+1}) = \min_\beta \int_0^{\tau_f} [\Theta(Q^k - \beta^k P^k) - Z]^2 \, d\tau. \tag{13}$$

This expression is linearized by Taylor expansion and then differentiated with respect to β^k. The following equation results for β^k

$$\beta^k = \left(\int_0^{\tau_f} (\Theta - Z) \, \Delta\Theta \, d\tau\right) \bigg/ \left(\int_0^{\tau_f} (\Delta\Theta)^2 \, d\tau\right). \tag{14}$$

If there were no measurement errors, the following stopping criteria could be used:

$$J(Q^{k+1}) < \text{a small number}. \tag{15}$$

However, in practical applications, measurement errors are always present; therefore the discrepancy principle [7, 8] should be used to establish the stopping criterion; that is assuming constant standard deviation of the measurement error σ, we have

$$\int_0^{\tau_f} \sigma^2 \, d\tau = \xi^2. \tag{16a}$$

Then the stopping criterion is taken as

$$J(Q^{k+1}) < \xi^2. \tag{16b}$$

SOLUTION ALGORITHM

The iterative computation algorithm for the solution of this inverse problem can be summarized as follows: suppose an initial guess is available for $Q^k(\tau)$ at iteration k.

Step 1. Solve the Direct problem given by equations (1), to obtain $\Theta(x, y, \tau)$; **Step 2**. Continue if the stopping criterion given by equations (16) is not satisfied; **Step 3**. Knowing $\Theta(x, y, \tau)$ and measured temperature $Z(x, y, \tau)$, solve the adjoint problem (6) and obtain adjoint variable $\lambda(x, y, \tau)$; **Step 4**. Knowing $\lambda(x, 1, \tau)$, compute $J'(\tau)$ from equation (9); **Step 5**. Knowing the gradient equation $J'(\tau)$, compute γ^k from equation (12) and the direction of descent P^k from equation (11); **Step 6**. Knowing P^k, solve the sensitivity problem (3) by setting $\Delta Q = P^k$ and obtain $\Delta\Theta(P^k)$; **Step 7**. Knowing $\Delta\Theta(P^k)$, compute step size β^k from equation (14); **Step 8**. Knowing step size β^k, compute new wall heat flux Q^{k+1} from equation (10); **Step 9**. Go to step 1.

RESULTS AND DISCUSSION

To evaluate the accuracy of the inverse analysis for estimating $Q(\tau)$, the simulated temperature data, Z, are generated by adding random errors $\omega\sigma$ to the exact temperatures, Θ, computed from the direct problem. Then, Z is expressed as

$$Z = \Theta + \omega\sigma, \tag{17}$$

where σ is the standard deviation of measurement errors which is assumed to be the same for all measurements, as shown in equation (16), and ω is the normally distributed random number generated by the IMSL subroutine DRNNOR [9]. For normally distributed random numbers, there is a 99% probability of the value of ω lying in the range

$$-2.576 < \omega < 2.576. \tag{18}$$

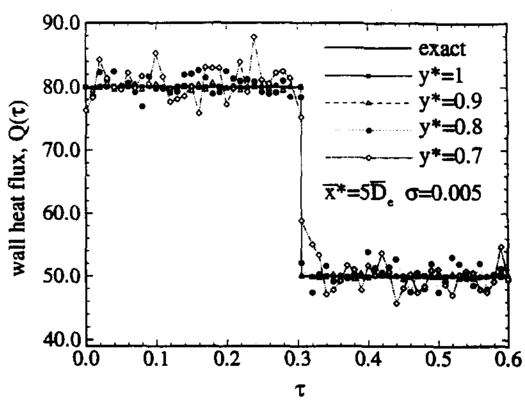
In present study, we investigate two different timewise variations of wall heat flux, $Q(\tau)$:

$$\text{Case (A)} \quad Q(\tau) = \begin{cases} 80 & 0 \leqslant \tau \leqslant 0.3 \\ 50 & 0.3 < \tau \leqslant 0.6 \end{cases} \tag{19}$$

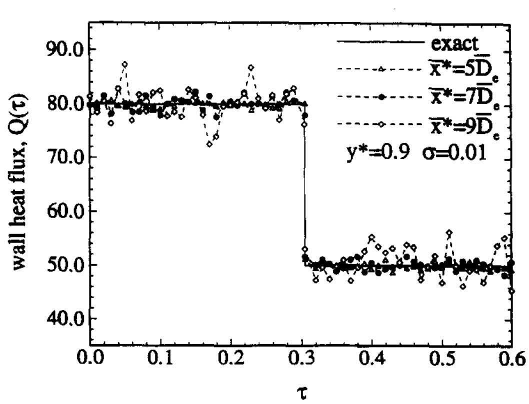
$$\text{Case (B)} \quad Q(\tau) = \begin{cases} 50 + 100\tau & 0 \leqslant \tau \leqslant 0.3 \\ 50 + 100(0.6 - \tau) & 0.3 < \tau \leqslant 0.6 \end{cases} \tag{20}$$

Effect of the sensor location

Figure 1(a) shows the effects of sensor location on the accuracy of estimations for a standard deviation of $\sigma = 0.005$, which corresponded to a 1.25% measurement error for the case of $Re = 10^5$ and $Pr = 1$. The three different



(1a)



(1b)

Fig. 1. (a) Effect of the transverse location, y^*, of the sensor on the accuracy of estimations. (b) Effect of the axial location, $\bar{x}^*$, of the sensor on the accuracy of estimations.

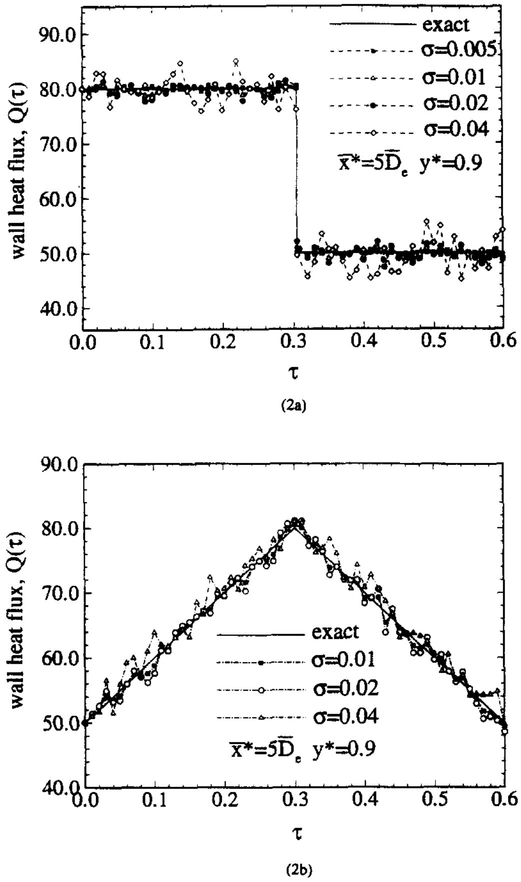


Fig. 2. (a) Effect of the measurement errors (standard deviation σ) on the accuracy of estimations for case A. (b) Effect of the measurement errors (standard deviation σ) on the accuracy of estimations for case B.

axial sensor locations, in dimensional form, include $\bar{x}^* = 5\bar{D}_e, 7\bar{D}_e, 9\bar{D}_e$ with $\bar{D}_e = 4$ in; and four different transverse sensor locations are taken as $\bar{y}^* = 1$, 0.9, 0.8, and 0.7 in for a duct having a half channel height of $\bar{h} = 1$ in (i.e. $\bar{D}_e = 4$ in). The dimensionless location $y^* = 1$ would correspond to a sensor at the wall. The accuracy of the estimation decreased with decreasing y^* (i.e. as the distance between the sensor junction and the wall increased). The sensor location $y^* < 0.7$ corresponds to the region outside the thermal boundary layer; as a result temperature measurements taken in such a region could not be used for inverse analysis.

Figure 1(b) shows the effect of axial location of the sensor on the accuracy of the estimation. In this figure we examine three axial locations ($\bar{x}^* = 5\bar{D}_e, 7\bar{D}_e$, and $9\bar{D}_e$) with the transverse position taken as $y^* = 0.9$ and the standard deviation $\sigma = 0.01$ (which corresponds to 2.5% measurement error). Clearly, increasing the axial location $\bar{x}^*$ of the sensor decreases the accuracy of the estimation, because the sensor location coincides with the fully developed region.

Effect of measurement error

Figure 2(a) illustrates the estimation made by using standard deviation of the measurement error taken as $\sigma = 0.005$, 0.01, 0.02 and 0.04 (corresponding to 1.25, 2.5, 5 and 10% error, respectively) for a sensor located at $\bar{x}^* = 5\bar{D}_e$, $y^* = 0.9$. As expected, increasing measurement error decreases the accuracy of estimation.

Effect of functional form of the wall heat flux

Figure 2(a, b) shows the accuracy of estimation for two different functional forms of the applied wall heat flux, $Q(\tau)$, for the sensor location of $\bar{x}^* = 5\bar{D}_e$, $y^* = 0.9$ and the standard deviation of measurement error $\sigma = 0.01$, 0.02 and 0.04. It appears that with the present method of analysis, estimation can be made up to about 10% measurement error.

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